Continuous Random Variables Lecture 3

Normal Distribution

Def: A continuous random variable Z is said to have the standard Normal distribution if its pdf φ is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad -\infty < z < \infty$$

To see that this is indeed a proper pdf observe that

$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$ as follows:

$$\text{Let } I = \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy.$$

$$\text{Therefore } I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx\,dy$$

The domain of integration is the entire plane $\mathbb{R}^2$.

We can express this integral in terms of polar coordinates

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty} r e^{-\frac{r^2}{2}} dr\right)$$

$$= 2\pi \left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} = 2\pi.$$

$$\text{Hence } I = \sqrt{I^2} = \sqrt{2\pi}$$

The standard Normal cdf Φ is the accumulated area under the pdf:

$$\Phi(z) = \int_{-\infty}^{z} \varphi(t)\,dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt$$

There are several important symmetry properties that can be deduced from the standard Normal pdf and cdf.

1. Symmetry of pdf: $\varphi(-z) = \varphi(z)$, i.e. φ is an even function.

2. Symmetry of tail areas: $\Phi(-z) = 1 - \Phi(z)$.

This can be readily seen by noting that if $-z$ is to the left of 0 $P(Z \leq -z) = P(Z \geq z) = 1 - P(Z < z)$.

Formally $\Phi(-z) = \int_{-\infty}^{-z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt \overset{u=-t}{=\!=}$

$$= \int_{\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-u)^2}{2}} (-du) = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du =$$

$$= 1 - \Phi(z).$$

3. Symmetry of Z and $-Z$: If Z has standard Normal distribution then so does $-Z$.

To see this, note that the cdf of $-Z$ is

$$P(-Z \le z) = P(Z \ge -z) = 1 - \Phi(-z) = \Phi(z).$$

Thus the cdf and pdf of $-Z$ are identical.

The expected value and variance of the standard Normal distribution are easy to compute.

$$E[Z] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} dz = 0 \quad \text{since } z e^{-\frac{z^2}{2}} \text{ is}$$

an odd function.

$$E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{z^2}{2}}$$

$$= \frac{2}{\sqrt{2\pi}} \left(-z e^{-\frac{z^2}{2}} \Big|_0^{\infty} + \int_0^{\infty} e^{-\frac{z^2}{2}} dz \right)$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} = 1.$$

Clearly $Var(Z) = E[Z^2] - (E[Z])^2 = E[Z^2] = 1.$

<u>Def:</u> If Z is tandard Normal then $X = \mu + \sigma Z$ is said to have the Normal distribution with mean μ and variance σ^2. Indeed,

$$E[X] = E[\mu + \sigma Z] = \mu + \sigma E[Z] = \mu$$

$$Var(X) = E[(X-\mu)^2] = E[(\sigma Z)^2] = \sigma^2 E[Z^2] = \sigma^2.$$

If X is normally distributed with mean μ and variance σ^2 then we can convert X into the standard Normal variable z by $z = \frac{X-\mu}{\sigma}$

Accordingly the cdf and pdf of X are given by

$$F(x) = P(X \leq x) = P(\mu + \sigma Z \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) = \varphi\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Thm: (68-95-99.7%) If X is Normal with mean μ and variance σ^2 then

$$P(|X-\mu| < \sigma) \approx 0.68$$

$$P(|X-\mu| < 2\sigma) \approx 0.95$$

$$P(|X-\mu| < 3\sigma) \approx 0.997.$$

Ex. Let X be normal with mean -1 and variance 4. What is $P(|X| < 3)$ exactly in terms of Φ and approximately?

<u>Solution:</u> $P(|X|<3) = P(-3<X<3) =$

$$= P\left(\frac{-3+1}{2} < \frac{X+1}{2} < \frac{3+1}{2}\right) = P(-1<Z<2)$$

$$= \varphi(2) - \varphi(-1) = \varphi(2) - (1-\varphi(1)) = \varphi(1)+\varphi(2)-1$$

We can approximate as follows:

$$P(-1<Z<1) + P(1<Z<2) = P(|Z|<1) +$$

$$+ \frac{P(|Z|<2) - P(|Z|<1)}{2} \approx 0.68 + \frac{0.95-0.68}{2} = 0.815$$

It is convenient to use computer software to estimate $\varphi(z)$ or an integration table in the absence of such software.

<u>Ex.</u> (Folded Normal) Let $Y=|Z|$ where Z is the standard Normal. Find the mean, variance, and the distribution of Y.

<u>Solution:</u> $E[Y] = \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz =$

$$= 2\int_0^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{2}{\sqrt{2\pi}}\left[-e^{-\frac{z^2}{2}}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}$$

Since $Y^2=Z^2$, $E[Y^2] = E[Z^2] = 1$.

Hence $Var(Y) = E[Y^2] - (E[Y])^2 = 1 - \frac{2}{\pi}$.

Finally,

$$F_Y(y) = P(Y \leq y) = P(|Z| \leq y) = P(-y \leq Z \leq y)$$

$$= \Phi(y) - \Phi(-y) = \Phi(y) - (1-\Phi(y)) = 2\Phi(y) - 1$$

and $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(2\Phi(y)-1\right) = 2\varphi(y)$

for $y \geq 0$.

<u>Ex.</u> If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$. Estimate

(a) $P(2 < X < 5)$.

(b) $P(X > 0)$

(c) $P(|X-3| > 6)$

<u>Solution:</u>

(a) $P(2 < X < 5) = P\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right) =$

$= P\left(-\frac{1}{3} < Z < \frac{2}{3}\right) = \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) = \Phi\left(\frac{1}{3}\right) + \Phi\left(\frac{2}{3}\right) - 1$

$\approx \Phi(0.33) + \Phi(0.67) - 1 \overset{\text{from table}}{\approx} 0.6293 + 0.7486 - 1$

$= 0.3779$

(b) $P(X > 0) = P(Z > -1) = 1 - \Phi(-1) = 1 - (1 - \Phi(1))$

$= \Phi(1) \approx 0.8413$

(c) $P(|X-3| > 6) = P\left((X-3) > 6\right) + P\left((X-3) < -6\right) =$

$= P(Z > 2) + P(Z < -2) = 2P(Z > 2) = 2(1 - \Phi(2)) \approx 0.0456.$